

# From Yang-Baxter maps to integrable quad maps and recurrences

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## Abstract

Starting from known solutions of the functional Yang-Baxter equations, we exhibit Miura type of transformations leading to various known integrable quad equations. We then construct, from the same list of Yang-Baxter maps, a series of non-autonomous solvable recurrences of order two.

## 1 Introduction

A number of discrete integrable systems have been described in the literature in the last decades. They are mostly discrete versions of known integrable differential equations and partial differential equations, retaining fundamental features of integrability, as for example the existence of a sufficient number of invariant tori and hierarchies of associated integrable equations. Important classification results [1] have been obtained, both for the ordinary difference equations case (in particular integrable maps [2] and discrete Painlevé equations [3]) and for the partial difference equations (integrable lattice equations and Yang-Baxter maps among others [4, 5, 6]).

We show that there exist bridges between different systems. We examine two such situations. Starting from Yang-Baxter maps, given as lattice equations on a quad graph with

variables on the bonds, we first construct (section 3), via Miura type of transformations, quad maps with variables on the vertices, establishing a correspondence between the classifications of [4] and [5]. We then construct, from the same list of Yang-Baxter maps a series of solvable recurrences (section 4).

## 2 The setting

The underlying geometry behind the models is a two dimensional square lattice with variables associated to the edges and/or to the vertices. Each elementary cell is labelled by a pair of integers  $(m, n)$ , which is the pair of coordinates of the lower left corner of the cell. Whenever possible we will use the standard notation “bar” and “tilde” to indicate shifts in the  $n$  and  $m$  direction respectively i.e.  $w \equiv w(m, n)$ ,  $\bar{w} \equiv w(m, n + 1)$ ,  $\tilde{w} \equiv w(m + 1, n)$  and  $\tilde{\tilde{w}} \equiv w(m + 1, n + 1)$ . The elementary cell is depicted in Figure 1.

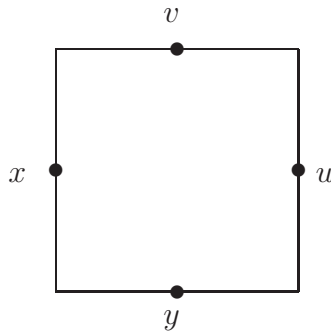


Figure 1: *Original cell*

The variables  $(x, y, u, v)$  are assigned to the bonds as shown in Figure 1. The models are defined by giving a  $2 \rightarrow 2$  map  $(x, y) \rightarrow (u, v)$ , verifying the functional Yang-Baxter equations (aka Yang-Baxter maps).

We start from the models described in [7, 6] and use the same labelling, namely the “F” and “H” families, which we reproduce here:

$$u = \alpha y P, \quad v = \beta x P, \quad P = \frac{(1 - \beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1 - \alpha)x + (\alpha - \beta)yx + \alpha(\beta - 1)y}, \quad (F_I)$$

$$u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y}, \quad (F_{II})$$

$$u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y}{x - y}, \quad (F_{III})$$

$$u = y P, \quad v = x P, \quad P = 1 + \frac{\beta - \alpha}{x - y}, \quad (F_{IV})$$

$$u = y + P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x - y}, \quad (F_V)$$

and

$$u = y Q^{-1}, \quad v = x Q, \quad Q = \frac{(1 - \beta)xy + (\beta - \alpha)y + \beta(\alpha - 1)}{(1 - \alpha)xy + (\alpha - \beta)x + \alpha(\beta - 1)}, \quad (H_I)$$

$$u = y Q^{-1}, \quad v = x Q, \quad Q = \frac{\alpha + (\beta - \alpha)y - \beta xy}{\beta + (\alpha - \beta)x - \alpha xy}, \quad (H_{II})$$

$$u = \frac{y}{\alpha}Q, \quad v = \frac{x}{\beta}Q, \quad Q = \frac{\alpha x + \beta y}{x + y}, \quad (H_{III}^A)$$

$$u = y Q^{-1}, \quad v = x Q, \quad Q = \frac{\alpha xy + 1}{\beta xy + 1}, \quad (H_{III}^B)$$

$$u = y - P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x + y}. \quad (H_V)$$

We will discard the cases  $H_{III}^A$  and  $H_V$  which are equivalent respectively to  $F_{III}$  and  $F_V$ , through the change  $u \rightarrow -u$  and  $v \rightarrow -v$ .

In section 3 we establish a correspondence between the original models and known integrable “quad equation”, that is to say partial difference equations, which happen to fall in a classification given by the same authors [5].

In section 4 we construct from the same family of models a number of solvable ordinary difference equations.

### 3 A correspondence via Miura transforms

We will bring in vertex variables  $(w, \tilde{w}, \bar{w}, \tilde{\bar{w}})$  (see Figure 2), subject to some local condition, via a Miura type of transformation of the original variables  $(x, y, u, v)$ .

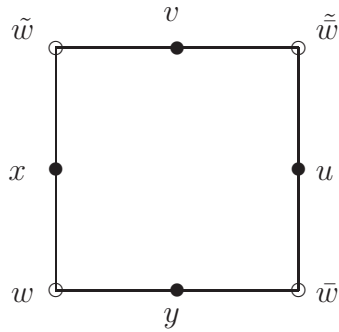


Figure 2: *Decorated cell*

#### 3.1 $F_I$

The first case we examine is  $F_I$ :

$$u = \alpha y P, \quad v = \beta x P, \quad P = \frac{(1 - \beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1 - \alpha)x + (\alpha - \beta)xy + \alpha(\beta - 1)y} \quad (1)$$

Define vertex variables  $(w, \tilde{w}, \bar{w}, \tilde{\tilde{w}})$  via the Miura transformation:

$$x = p w \tilde{w}, \quad u = p \bar{w} \tilde{\tilde{w}}, \quad v = q w \bar{w}, \quad y = q \tilde{w} \tilde{\tilde{w}} \quad (2)$$

It is straightforward to eliminate the  $x, y, v, u$  variables using Eq. (1) and obtain an equation for  $w$  alone. Setting  $\alpha = p^2$  and  $\beta = q^2$ , we obtain exactly equation A2 of [5]:

$$(q^2 - p^2)(w \tilde{w} \bar{w} \tilde{\tilde{w}} + 1) + p(1 - q^2)(w \tilde{w} + \bar{w} \tilde{\tilde{w}}) - q(1 - p^2)(w \bar{w} + \tilde{w} \tilde{\tilde{w}}) = 0 \quad (3)$$

in canonical form.

### 3.2 $F_{II}$

In the case of  $F_{II}$

$$u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y} \quad (4)$$

we introduce the Miura transformation:

$$x = p w \tilde{w}, \quad u = p \bar{w} \tilde{\tilde{w}}, \quad v = q w \bar{w}, \quad y = q \tilde{w} \tilde{\tilde{w}} \quad (5)$$

with  $q^2/p^2 = \alpha/\beta$ . We get

$$q (w \tilde{w} + \bar{w} \tilde{\tilde{w}}) - p (w \bar{w} + \tilde{w} \tilde{\tilde{w}}) + \frac{p}{q} - \frac{q}{p} = 0 \quad (6)$$

which is equation H3 of [5].

### 3.3 $F_{III}$

For system  $F_{III}$  we start from

$$u = \frac{y}{\alpha} P, \quad v = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y}{x - y} \quad (7)$$

and introduce the Miura transformation

$$x = p w \tilde{w}, \quad u = p \bar{w} \tilde{\tilde{w}}, \quad v = q w \bar{w}, \quad y = q \tilde{w} \tilde{\tilde{w}} \quad (8)$$

Substituting into the two equations we find the condition  $q^2/p^2 = \alpha/\beta$  whereupon  $w$  satisfies the equation

$$q (w \tilde{w} + \bar{w} \tilde{\tilde{w}}) - p (w \bar{w} + \tilde{w} \tilde{\tilde{w}}) = 0 \quad (9)$$

This is precisely the H3 equation of [5] for the special value  $\delta = 0$ . A more familiar interpretation of this equation is possible. Indeed solving for  $\tilde{\tilde{w}}$  we obtain

$$\tilde{\tilde{w}} = w \frac{q \tilde{w} - p \bar{w}}{p \tilde{w} - q \bar{w}} \quad (10)$$

which is just the potential form of the discrete modified KdV equation.

### 3.4 $F_{IV}$

The defining equation is

$$u = y P, \quad v = x P, \quad P = 1 - \frac{\alpha - \beta}{x - y} \quad (11)$$

Introduce the Miura transformations

$$x = p + w + \tilde{w}, \quad u = p + \bar{w} + \tilde{\bar{w}}, \quad v = q + w + \bar{w}, \quad y = q + \tilde{w} + \tilde{\bar{w}} \quad (12)$$

where  $\beta - \alpha = 2(p - q)$ . Eliminating the  $x, y, v, u$  variables using Eq. (11) we obtain an equation for  $w$  alone

$$(\tilde{\bar{w}} - w)(\tilde{w} - \bar{w}) + (p - q)(\tilde{\bar{w}} + \tilde{w} + \bar{w} + w) + p^2 - q^2 = 0 \quad (13)$$

Equation (13) is precisely equation H2 of [5].

### 3.5 $F_V$

The defining equation is

$$u = y + P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x - y} \quad (14)$$

We introduce the  $w$  variable by

$$x = w + \tilde{w}, \quad u = \bar{w} + \tilde{\bar{w}}, \quad v = w + \bar{w}, \quad y = \tilde{w} + \tilde{\bar{w}}. \quad (15)$$

It is straightforward to eliminate the  $x, y, v, u$  variables using Eq. (14) and obtain an equation for  $w$  alone. We find

$$(\tilde{\bar{w}} - w)(\bar{w} - \tilde{w}) = \alpha - \beta \quad (16)$$

which is the potential form of the discrete KdV equation that is to say equation H1 of [5].

Next we turn to the study of the  $H$  family.

### 3.6 $H_I$

The first case we examine is  $H_I$

$$u = y Q^{-1}, \quad v = x Q, \quad Q = \frac{(1 - \beta)xy + (\beta - \alpha)y + \beta(\alpha - 1)}{(1 - \alpha)xy + (\alpha - \beta)x + \alpha(\beta - 1)} \quad (17)$$

The Miura transformation is

$$x = p w \tilde{w}, \quad u = p (\bar{w} \tilde{\bar{w}})^{-1}, \quad v = q w \bar{w}, \quad y = q (\tilde{w} \tilde{\bar{w}})^{-1} \quad (18)$$

A direct substitution into Eq. (17) leads to an equation which does not have the canonical A2 form. However introducing  $\alpha = p^2$  and  $\beta = q^2$  we obtain

$$(q^2 - p^2) (w \tilde{w} \bar{w} \tilde{\bar{w}} + 1) + p(1 - q^2) (w \tilde{w} + \bar{w} \tilde{\bar{w}}) - q(1 - p^2)(w \bar{w} + \tilde{w} \tilde{\bar{w}}) = 0 \quad (19)$$

which is A2 in canonical form.

### 3.7 $H_{\text{II}}$

The defining relation is

$$u = y Q^{-1}, \quad v = x Q, \quad Q = \frac{\alpha + (\beta - \alpha)y - \beta xy}{\beta + (\alpha - \beta)x - \alpha xy} \quad (20)$$

The Miura transformation is

$$x = w\tilde{w}, \quad u = (\bar{w}\tilde{\bar{w}})^{-1}, \quad v = w\bar{w}, \quad y = (\tilde{w}\tilde{\bar{w}})^{-1} \quad (21)$$

and substituting into Eq. (20) we obtain

$$(\alpha - \beta)(w\tilde{w}\bar{w}\tilde{\bar{w}} + 1) + \beta(w\tilde{w} + \bar{w}\tilde{\bar{w}}) - \alpha(w\bar{w} + \tilde{w}\tilde{\bar{w}}) = 0 \quad (22)$$

This equation could be interpreted as a special form of the A2 equation of [5]. Since Eq. (3) has the generic form of A2 it is convenient to start from this form and put  $q = 1 + \epsilon\alpha$ ,  $p = 1 + \epsilon\beta$ . Taking the limit  $\epsilon \rightarrow 0$  we find precisely Eq. (22) which establishes its relation to A2.

### 3.8 $H_{\text{III}}^{\text{B}}$

We start from

$$u = \frac{y}{Q}, \quad v = x Q, \quad Q = \frac{\alpha xy + 1}{\beta xy + 1} \quad (23)$$

The Miura transformation is now

$$x = p w\tilde{w}, \quad u = -p (\bar{w}\tilde{\bar{w}})^{-1}, \quad v = q w\bar{w}, \quad y = -q (\tilde{w}\tilde{\bar{w}})^{-1} \quad (24)$$

Eliminating  $x, y, v, u$  and introducing  $\alpha p^2 = 1$ ,  $\beta q^2 = 1$  we obtain the equation

$$q (w\tilde{w} + \bar{w}\tilde{\bar{w}}) - p (w\bar{w} + \tilde{w}\tilde{\bar{w}}) = 0 \quad (25)$$

which is equation H3 of [5] for the special value  $\delta = 0$ .

We have thus established correspondences via appropriate Miura transformations, between the two classifications of [7, 6] and [5].

Remark: All these constructions can be made non-autonomous [8, 9, 10].

## 4 A second dynamical interpretation

One may view the original lattice in a different way: the bond variables may be assigned to the vertices of a finer square lattice, as shown in Figure 3.

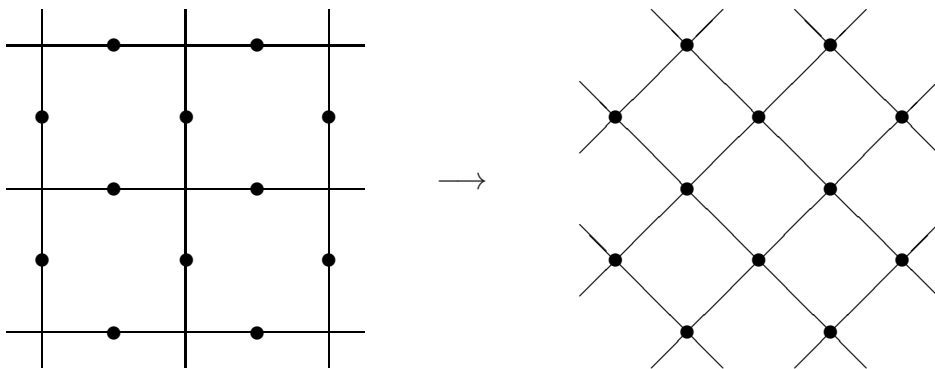


Figure 3: A change of point of view

We get a new square lattice whose elementary square cell is, after rotation of  $-\frac{\pi}{4}$ :

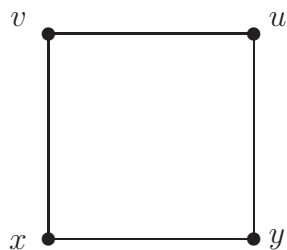


Figure 4: *New cell*

For all the models described in [7] the conditions relating  $u, v, x, y$  allow to define various rational  $2 \times 2$  maps: they give  $u$  and  $v$  in terms of  $x$  and  $y$ , but also  $x$  and  $y$  in terms of  $u$  and  $v$ , as well as  $u$  and  $y$  in terms of  $v$  and  $x$ , or  $v$  and  $x$  in terms of  $u$  and  $y$ . This is the “quadrirationality” property. We obtain in such a way four  $2 \times 2$  maps, defined for each elementary cell. These cells form a new two-dimensional lattice. Notice that there are two kinds of cells in this lattice. Half of them contained a vertex of the original lattice, and the other half did not. In the original lattice, we had relations only for the latter kind of cells. *We will impose the existence of relations as above for all the cells, allowing non constant parameters.*

This will lead to a possibly non autonomous recurrence of order two. The variables, denoted by  $x_{m,n}$ , are assigned to the vertices of the new lattice. Since on each cell we have four  $(m, n)$  dependent  $2 \times 2$  maps, we may, once given initial conditions  $x_{0,0}$  and  $x_{0,1}$ , calculate  $x$  over the whole lattice, provided some compatibility (zero monodromy around each vertex) condition is fulfilled.

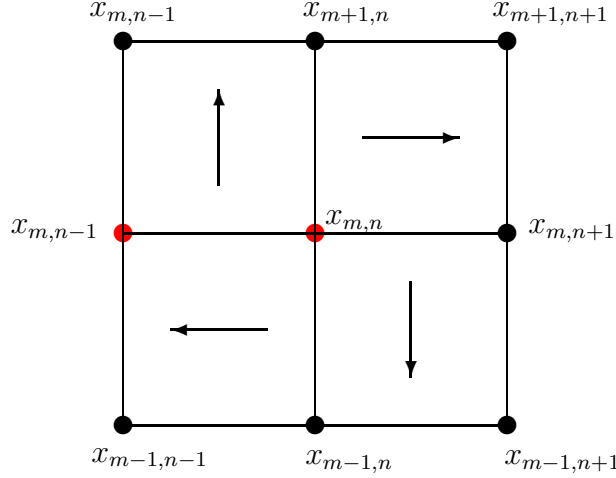


Figure 5: Local compatibility condition around vertex  $(m, n)$

We use the coordinates of the lower left corner to label the cells. Starting from  $x_{m,n}$  and  $x_{m,n-1}$ , we compute  $x_{m,n-1}$  and  $x_{m+1,n}$  using the same  $F$  and  $H$  equations as in the previous section, with  $(m, n)$  dependent set of parameters  $\Gamma_{m,n}$ . Going around the vertex  $(m, n)$  we calculate the surrounding values of  $x$ . This yields after four steps a value for  $x_{m,n-1}$  which has to coincide with the initial condition.

Remark: Doing so we reduce the dimensionality of the system, going from partial difference equation to ordinary difference equation. The reason is that we impose relations between variables which were not related in the original model, actually exactly on half of the cells of the new lattice.

In what follows, we shall show that it is indeed possible to satisfy the compatibility constraint. Moreover the resulting equations are explicitly solvable ones and given that the compatibility constraints are also explicitly solvable this allows us to give the solution for all  $x$ 's in closed form.

#### 4.1 $F_I$

In the case of  $F_I$ , the starting equations are

$$u = \alpha y P, \quad v = \beta P, \quad P = \frac{(1 - \beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1 - \alpha)x + (\alpha - \beta)xy + \alpha(\beta - 1)y}$$

which we rewrite as

$$x_{m+1,n+1} = \alpha_{m,n} x_{m,n+1} \frac{(1 - \beta_{m,n})x_{m,n} + \beta_{m,n} - \alpha_{m,n} + (\alpha_{m,n} - 1)x_{m,n+1}}{\beta_{m,n}(1 - \alpha_{m,n})x_{m,n} + (\alpha_{m,n} - \beta_{m,n})x_{m,n}x_{m,n+1} + \alpha_{m,n}(\beta_{m,n} - 1)x_{m,n+1}}$$

$$x_{m+1,n} = \beta_{m,n} x_{m,n} \frac{(1 - \beta_{m,n})x_{m,n} + \beta_{m,n} - \alpha_{m,n} + (\alpha_{m,n} - 1)x_{m,n+1}}{\beta_{m,n}(1 - \alpha_{m,n})x_{m,n} + (\alpha_{m,n} - \beta_{m,n})x_{m,n}x_{m,n+1} + \alpha_{m,n}(\beta_{m,n} - 1)x_{m,n+1}}$$



The compatibility conditions for the  $\alpha$  and  $\beta$  are

$$\alpha_{m+1,n+1} = \alpha_{m,n}, \quad \beta_{m+1,n} = \beta_{m,n+1} \quad (26)$$

and

$$\alpha_{m+1,n} = \beta_{m,n}\beta_{m,n+1}R, \quad \beta_{m+1,n+1} = \alpha_{m,n}\alpha_{m,n+1}R \quad (27)$$

where

$$R = \frac{(\alpha_{m,n} - 1)(\alpha_{m,n+1} - 1) - (\beta_{m,n} - 1)(\beta_{m,n+1} - 1)}{\alpha_{m,n}\alpha_{m,n+1}(\beta_{m,n} + \beta_{m,n+1} - 1) - \beta_{m,n}\beta_{m,n+1}(\alpha_{m,n} + \alpha_{m,n+1} - 1)} \quad (28)$$

The set of compatibility equations is itself overdetermined but it turns out that it is consistent and the solution is periodic in both  $m$  and  $n$ . We have

$$\alpha_{3m,3n} = \alpha_{3m+1,3n+1} = \alpha_{3m+2,3n+2} = \alpha_0$$

$$\alpha_{3m,3n+1} = \alpha_{3m+1,3n+2} = \alpha_{3m+2,3n} = \alpha_1$$

$$\alpha_{3m,3n+2} = \alpha_{3m+1,3n} = \alpha_{3m+2,3n+1} = \alpha_2$$

$$\beta_{3m,3n} = \beta_{3m+2,3n+1} = \beta_{3m+1,3n+2} = \beta_0$$

$$\beta_{3m,3n+1} = \beta_{3m+2,3n+2} = \beta_{3m+1,3n} = \beta_1$$

$$\beta_{3m,3n+2} = \beta_{3m+2,3n} = \beta_{3m+1,3n+1} = \beta_2$$

where  $\alpha_2, \beta_2$  are given in terms of  $\alpha_0, \alpha_1, \beta_0, \beta_1$  using Eq. (27). The solution for  $x_{m,n}$  is equally periodic, with period 3 in both  $m$  and  $n$  directions. In this case however all nine  $x_{3m+j,3n+k}$  for  $j, k = 0, 1, 2$  are generically different.

## 4.2 $F_{\text{II}}$

We start from the equations

$$u = \frac{y}{\alpha}P, \quad v = \frac{x}{\beta}P, \quad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y} \quad (29)$$

which we rewrite, introducing  $\gamma = \alpha/\beta$ , as

$$\begin{aligned} x_{m+1,n+1} &= x_{m,n+1} \left( \frac{x_{m,n} - x_{m,n+1}/\gamma_{m,n} + 1 - 1/\gamma_{m,n}}{x_{m,n} - x_{m,n+1}} \right), \\ x_{m+1,n} &= x_{m,n} \left( \frac{\gamma_{m,n}x_{m,n} - x_{m,n+1} + \gamma_{m,n} - 1}{x_{m,n} - x_{m,n+1}} \right) \end{aligned} \quad (30)$$

The compatibility condition is now

$$\gamma_{m+1,n+1}\gamma_{m+1,n}\gamma_{m,n+1}\gamma_{m,n} - \gamma_{m+1,n}\gamma_{m,n+1}\gamma_{m,n} - \gamma_{m+1,n+1}\gamma_{m+1,n}\gamma_{m,n+1} + \gamma_{m+1,n} + \gamma_{m,n+1} - 1 = 0$$

which can be rewritten as

$$(1 - \gamma_{m+1,n+1})(1 - \gamma_{m,n}) = (1 - 1/\gamma_{m+1,n})(1 - 1/\gamma_{m,n+1}) \quad (31)$$

which can be recognized as the linearisable equation of Hydon-Viallet (Eq (22) in [11], originally given in terms of the variable  $z = 1 - \gamma$ ). The solution to Eq. (31) can be obtained most easily if we introduce an auxiliary variable  $\chi$  and express  $\gamma$  as

$$\gamma_{m,n} = \frac{(1 - \chi_{m+1,n+1})(1 - \chi_{m,n})}{(1 - \chi_{m+1,n})(1 - \chi_{m,n+1})} \quad (32)$$

with  $\chi$  satisfying the equation

$$\chi_{m+1,n+1}\chi_{m,n} = \chi_{m+1,n}\chi_{m,n+1} \quad (33)$$

i.e.  $\chi$  is the product of two free functions  $\chi_{m,n} = f(m)g(n)$ . Using this solution we can obtain the general solution for  $x$  as

$$x_{m,n} = \frac{af(m)g(n) + bf(m) + cg(n) + a + 1}{1 - f(m)g(n)} \quad (34)$$

where  $a, b, c$  are constants related by  $a(a + 1) = bc$ .

### 4.3 $F_{\text{III}}$

In the case of  $F_{\text{III}}$ , where

$$u = \frac{y}{\alpha}P, \quad v = \frac{x}{\beta}P, \quad P = \frac{\alpha x - \beta y}{x - y} \quad (35)$$

we introduce the parameter  $\gamma = \alpha/\beta$ , and rewrite the equations as

$$x_{m+1,n+1} = x_{m,n+1} \left( \frac{x_{m,n} - x_{m,n+1}/\gamma_{m,n}}{x_{m,n} - x_{m,n+1}} \right), \quad x_{m+1,n} = x_{m,n} \left( \frac{\gamma_{m,n}x_{m,n} - x_{m,n+1}}{x_{m,n} - x_{m,n+1}} \right) \quad (36)$$

The compatibility condition is exactly the same as before, namely  $\gamma$  satisfies equation (31). In this case, it is more convenient to express  $\gamma$  in terms of an auxiliary variable  $\omega$  as

$$\gamma_{m,n} = \frac{\omega_{m+1,n+1}\omega_{m,n}}{\omega_{m+1,n}\omega_{m,n+1}} \quad (37)$$

and take  $\omega$  satisfying the equation

$$\omega_{m+1,n+1} + \omega_{m,n} = \omega_{m+1,n} + \omega_{m,n+1} \quad (38)$$

i.e.  $\omega$  is the sum of two free functions  $\omega_{m,n} = f(m) + g(n)$  which leads to the solution for  $\gamma$

$$\gamma_{m,n} = \frac{(f(m+1) + g(n+1))(f(m) + g(n))}{(f(m+1) + g(n))(f(m) + g(n+1))} \quad (39)$$

The general solution for  $x$  can now be given

$$x_{m,n} = a \frac{(f(m) + b)(g(n) - b)}{f(m) + g(n)} \quad (40)$$

where  $a, b$  are constants.

## 4.4 $F_{IV}$

The defining equations

$$u = y P, \quad v = x P, \quad P = 1 - \frac{\alpha - \beta}{x - y} \quad (41)$$

are rewritten as:

$$x_{m+1,n+1} = x_{m,n+1} \left( 1 - \frac{\gamma_{m,n}}{x_{m,n} - x_{m,n+1}} \right), \quad x_{m+1,n} = x_{m,n} \left( 1 - \frac{\gamma_{m,n}}{x_{m,n} - x_{m,n+1}} \right) \quad (42)$$

where  $\gamma = \alpha - \beta$ .

The compatibility condition in this case turns out to be

$$\gamma_{m+1,n+1} \gamma_{m,n} = \gamma_{m+1,n} \gamma_{m,n+1} \quad (43)$$

the solution of which is  $\gamma_{m,n} = \phi(m)\psi(n)$ , where  $\phi$  and  $\psi$  are free functions of their argument. At this point it is convenient to introduce two functions  $f(m)$  and  $g(n)$  such that  $f(m+1) - f(m) = \phi(m)$  and  $g(n+1) - g(n) = \psi(n)$ . The general solution here can be written as

$$x_{m,n} = (f(m) + a)(g(n) + b) \quad (44)$$

where  $a, b$  are constants.

## 4.5 $F_V$

The defining equations are

$$u = y + P, \quad v = x + P, \quad P = \frac{\gamma}{x - y} \quad (45)$$

with the same definition of  $\gamma$  as before, and we rewrite them as

$$x_{m+1,n+1} = x_{m,n+1} + \frac{\gamma_{m,n}}{x_{m,n} - x_{m,n+1}}, \quad x_{m+1,n} = x_{m,n} + \frac{\gamma_{m,n}}{x_{m,n} - x_{m,n+1}} \quad (46)$$

The compatibility condition in this case turns out to be the same as in the previous case, namely Eq. (43). Using the same functions  $f$  and  $g$  as above, it is straightforward to verify that

$$x_{m,n} = af(m) + \frac{g(n)}{a} + b \quad (47)$$

is the general solution of system (46), where  $a, b$  are constants.

Next we turn to the study of the  $H$  family.

## 4.6 $H_I$

The first case to examine is  $H_I$

$$u = yQ^{-1}, \quad v = xQ, \quad Q = \frac{(1 - \beta)xy + (\beta - \alpha)y + \beta(\alpha - 1)}{(1 - \alpha)xy + (\alpha - \beta)x + \alpha(\beta - 1)} \quad (48)$$

which we rewrite, after a translation  $\alpha = \gamma + 1$ ,  $\beta = \delta + 1$ , as

$$x_{m+1,n+1} = x_{m,n+1} \frac{\gamma_{m,n}x_{m,n}x_{m,n+1} - (\gamma_{m,n} - \delta_{m,n})x_{m,n} - \delta_{m,n}(\gamma_{m,n} + 1)}{\delta_{m,n}x_{m,n}x_{m,n+1} + (\gamma_{m,n} - \delta_{m,n})x_{m,n+1} - \gamma_{m,n}(\delta_{m,n} + 1)} \quad (49)$$

$$x_{m+1,n} = x_{m,n} \frac{\delta_{m,n}x_{m,n}x_{m,n+1} + (\gamma_{m,n} - \delta_{m,n})x_{m,n+1} - \gamma_{m,n}(\delta_{m,n} + 1)}{\gamma_{m,n}x_{m,n}x_{m,n+1} - (\gamma_{m,n} - \delta_{m,n})x_{m,n} - \delta_{m,n}(\gamma_{m,n} + 1)} \quad (50)$$

In this case the two parameters  $\gamma$  and  $\delta$  cannot be combined into a single parameter. The compatibility condition turns out to be

$$\gamma_{m+1,n} = \delta_{m,n}, \quad \delta_{m+1,n} = \gamma_{m,n} \quad (51)$$

which means that  $\gamma$  and  $\delta$  can be expressed in terms of two free functions  $f, g$  of  $n$  alone:  $\gamma_{2m,n} = \delta_{2m+1,n} = f(n)$ ,  $\gamma_{2m+1,n} = \delta_{2m,n} = g(n)$ . The solution of Eqs. (49), (50) is also periodic in the  $m$ -direction:  $x_{m+2,n} = x_{m,n}$ . Thus the equation becomes a one-dimensional one for the variable  $x$  at given parity of  $m$ . Once we have obtained the solution for one parity the solution at the other parity is obtained through Eq. (49) or (50). Taking the index  $m$  to be even we obtain the equation for  $x$

$$\begin{aligned} & (g(n)g(n-1) - f(n)f(n-1))x_{n-1}x_nx_{n+1} + \\ & + g(n)(f(n-1) - g(n-1))x_nx_{n+1} + f(n-1)(f(n) - g(n))x_nx_{n-1} + \\ & + f(n-1)(g(n-1) + 1)(f(n) - g(n))x_{n+1} + g(n)(f(n) + 1)(f(n-1) - g(n-1))x_{n-1} + \\ & + g(n)g(n-1)(f(n) + f(n-1) + 1) - f(n)f(n-1)(g(n) + g(n-1) + 1) = 0 \end{aligned} \quad (52)$$

where we have dropped the even index  $m$ . (In fact the equation for odd- $m$  is the same up to the permutation of  $f$  and  $g$ ). Equation (52) is linearisable. A study of the growth of the degrees of the iterates on an initial condition, using algebraic entropy techniques, leads to the following sequence of degrees: 0,1,1,2,2,3,3,4,4,5, ... This suggest that the mapping defined by (52) is of Gambier type [12] and this turns out to be indeed the case. We find the invariant

$$K = (-1)^n \frac{f(n)(x_n + 1)(x_{n+1} - 1) + g(n)(x_n - 1)(x_{n+1} + 1) - 2f(n)g(n)}{(f(n) - g(n))(x_n - 1)(x_{n+1} - 1)} \quad (53)$$

which allows to reduce Eq. (52) to a homographic mapping.

## 4.7 $H_{II}$

The defining equations

$$u = yQ^{-1}, \quad v = xQ, \quad Q = \frac{\alpha + (\beta - \alpha)y - \beta xy}{\beta + (\alpha - \beta)x - \alpha xy} \quad (54)$$

can be cast into the form

$$x_{m+1,n+1} = x_{m,n+1} \frac{1 + (\gamma_{m,n} - 1)x_{m,n} - \gamma_{m,n}x_{m,n}x_{m,n+1}}{\gamma_{m,n} + (1 - \gamma_{m,n})x_{m,n+1} - x_{m,n}x_{m,n+1}} \quad (55)$$

$$x_{m+1,n} = x_{m,n} \frac{\gamma_{m,n} + (1 - \gamma_{m,n})x_{m,n+1} - x_{m,n}x_{m,n+1}}{1 + (\gamma_{m,n} - 1)x_{m,n} - \gamma_{m,n}x_{m,n}x_{m,n+1}} \quad (56)$$

where  $\gamma = \alpha/\beta$ . It turns out that the compatibility condition for  $\gamma$  is precisely Eq. (31) i.e. the Hydon-Viallet equation. We give the solution for  $\gamma$  in the same form as in the case of  $F_{II}$ , namely

$$\gamma_{m,n} = \frac{(1 - f(n+1)g(m+1))(1 - f(n)g(m))}{(1 - f(n+1)g(m))(1 - f(n)g(m+1))} \quad (57)$$

We can now give the solution for  $x$  which turns out to possess an even-odd freedom

$$x_{m,n} = \frac{(1 + a_n f(n))(g(m) + a_{n+1})}{(1 + a_{n+1} f(n))(g(m) + a_n)} \quad (58)$$

where  $a_n$  is a period-two constant, i.e.  $a_{n+2} = a_n$ .

## 4.8 $H_{III}^B$

The set of equations  $H_{III}^B$

$$u = \frac{y}{Q}, \quad v = x Q, \quad Q = \frac{\alpha xy + 1}{\beta xy + 1} \quad (59)$$

yield

$$x_{m+1,n+1} = x_{m,n+1} \frac{\beta_{m,n}x_{m,n}x_{m,n+1} + 1}{\alpha_{m,n}x_{m,n}x_{m,n+1} + 1}, \quad x_{m+1,n} = x_{m,n} \frac{\alpha_{m,n}x_{m,n}x_{m,n+1} + 1}{\beta_{m,n}x_{m,n}x_{m,n+1} + 1} \quad (60)$$

The latter can be written in more compact form if we introduce  $\alpha = \gamma + \delta$  and  $\beta = \gamma - \delta$ . We obtain

$$\delta_{m,n}\delta_{m+1,n+1} = \delta_{m,n+1}\delta_{m+1,n} \quad (61)$$

$$(\gamma_{m+1,n} - \gamma_{m,n})\delta_{m+1,n+1} + (\gamma_{m+1,n+1} - \gamma_{m,n+1})\delta_{m+1,n} = 0 \quad (62)$$

which can be readily integrated to

$$\alpha_{m,n} = f(n)(h(n) + (-1)^n k(m) + g(m)) \quad (63)$$

$$\beta_{m,n} = f(n)(h(n) + (-1)^n k(m) - g(m)) \quad (64)$$

Next we remark that from Eq. (60) we have

$$x_{m,n}x_{m,n+1} = x_{m+1,n}x_{m+1,n+1} \equiv \phi(n) \quad (65)$$

where  $\phi$  is, obviously, independent of  $m$ . Given the expressions of  $\alpha$  and  $\beta$  we can obtain the value of  $1/\phi = f(n)(-h(n) + (-1)^n c)$ , where  $c$  is an integration constant, the value of which is fixed by the initial conditions. The values of  $x$  can now be obtained from the recursion

$$x_{m,n+2} = \frac{\phi(n+1)}{\phi(n)} x_{m,n} \quad (66)$$

Using the expression for  $\phi$  we can, starting from some initial condition  $x_{0,0}, x_{0,1}$  obtain all  $x_{0,n}$  for all  $n$ . Once the  $x_{0,n}$  are obtained we can compute  $x_{m,n}$  by

$$x_{m+1,2n} = \frac{c - k(m) - g(m)}{c - k(m) + g(m)} x_{m,2n} \quad (67)$$

$$x_{m+1,2n+1} = \frac{c - k(m) + g(m)}{c - k(m) - g(m)} x_{m,2n+1} \quad (68)$$

Thus the solution to Eq. (60) can be expressed as a double semi-infinite product.

## 5 Conclusion

Integrable discrete equations appear in an innumerable number of avatars, and we have exhibited some relations between such forms.

The relations we have described enter the general problem of classifying integrable discrete equations. One should nevertheless emphasize that the correspondences we have established are not invertible. They consequently do not define equivalences between models.

This is reminiscent of what happens when one goes from a discrete equation to its “potential form”, or when one performs reductions by symmetries. These are other types of transformations establishing relations between different integrable systems, and the links between these various correspondences and the one we presented here still need to be explored.

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